

Separability of Dirac equation in higher dimensional Kerr-NUT-de Sitter spacetime

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Abstract

It is shown that the Dirac equations in general higher dimensional Kerr-NUT-de Sitter spacetimes are separated into ordinary differential equations.

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Recently, the separability of Klein-Gordon equations in higher dimensional Kerr-NUT-de Sitter spacetimes [1] was shown by Frolov, Krtouš and Kubizňák [2]. This separation is deeply related to that of geodesic Hamilton-Jacobi equations. Indeed, a geometrical object called conformal Killing-Yano tensor plays an important role in the separability theory [3, 4, 5, 2, 6, 7, 8, 9]. However, at present, a similar separation of the variables of Dirac equations is lacking, although the separability in the four dimensional Kerr geometry was given by Chandrasekhar [10]. In this paper we shall show that Dirac equations can also be separated in general Kerr-NUT-de Sitter spacetimes.

The D -dimensional Kerr-NUT-de Sitter metrics are written as follows [1]:

(a) $D = 2n$

$$g^{(2n)} = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu(x)} + \sum_{\mu=1}^n Q_\mu(x) \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2, \quad (1)$$

(b) $D = 2n + 1$

$$g^{(2n+1)} = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu(x)} + \sum_{\mu=1}^n Q_\mu(x) \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 + \frac{c}{A^{(n)}} \left(\sum_{k=0}^n A^{(k)} d\psi_k \right)^2. \quad (2)$$

The functions Q_μ ($\mu = 1, 2, \dots, n$) are given by

$$Q_\mu(x) = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (x_\mu^2 - x_\nu^2), \quad (3)$$

where X_μ is a function depending only on the coordinate x_μ , and $A^{(k)}$ and $A_\mu^{(k)}$ are the elementary symmetric functions of $\{x_\nu^2\}$ and $\{x_\nu^2\}_{\nu \neq \mu}$ respectively:

$$\prod_{\nu=1}^n (t - x_\nu^2) = A^{(0)} t^n - A^{(1)} t^{n-1} + \dots + (-1)^n A^{(n)}, \quad (4)$$

$$\prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (t - x_\nu^2) = A_\mu^{(0)} t^{n-1} - A_\mu^{(1)} t^{n-2} + \dots + (-1)^{n-1} A_\mu^{(n-1)}. \quad (5)$$

The metrics are Einstein if X_μ takes the form [1, 11]

(a) $D = 2n$

$$X_\mu = \sum_{k=0}^n c_{2k} x_\mu^{2k} + b_\mu x_\mu, \quad (6)$$

(b) $D = 2n + 1$

$$X_\mu = \sum_{k=0}^n c_{2k} x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}, \quad (7)$$

where c, c_{2k} and b_μ are free parameters.

1. D=2n

For the metric (1) we introduce the following orthonormal basis $\{e^a\} = \{e^\mu, e^{n+\mu}\}$ ($\mu = 1, 2, \dots, n$):

$$e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad e^{n+\mu} = \sqrt{Q_\mu} \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k. \quad (8)$$

The dual vector fields are given by

$$e_\mu = \sqrt{Q_\mu} \frac{\partial}{\partial x_\mu}, \quad e_{n+\mu} = \sum_{k=0}^{n-1} \frac{(-1)^k x_\mu^{2(n-1-k)}}{\sqrt{Q_\mu} U_\mu} \frac{\partial}{\partial \psi_k}. \quad (9)$$

The spin connection is calculated as [11]

$$\begin{aligned}
\omega_{\mu\nu} &= -\frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2-x_\nu^2}e^\mu - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2-x_\nu^2}e^\nu, \quad (\mu \neq \nu) \\
\omega_{\mu,n+\mu} &= -(\partial_\mu\sqrt{Q_\mu})e^{n+\mu} - \sum_{\rho \neq \mu} \frac{x_\mu\sqrt{Q_\rho}}{x_\rho^2-x_\mu^2}e^{n+\rho}, \quad (\text{no sum over } \mu), \\
\omega_{\mu,n+\nu} &= \frac{x_\mu\sqrt{Q_\nu}}{x_\mu^2-x_\nu^2}e^{n+\mu} - \frac{x_\nu\sqrt{Q_\mu}}{x_\mu^2-x_\nu^2}e^{n+\nu}, \quad (\mu \neq \nu) \\
\omega_{n+\mu,n+\nu} &= -\frac{x_\mu\sqrt{Q_\nu}}{x_\mu^2-x_\nu^2}e^\mu - \frac{x_\nu\sqrt{Q_\mu}}{x_\mu^2-x_\nu^2}e^\nu, \quad (\mu \neq \nu).
\end{aligned} \tag{10}$$

Then, the Dirac equation is written in the form

$$(\gamma^a D_a + m)\Psi = 0, \tag{11}$$

where D_a is a covariant differentiation,

$$D_a = e_a + \frac{1}{4}\omega_{bc}(e_a)\gamma^b\gamma^c. \tag{12}$$

From (9),(10) and (12), we obtain the explicit expression for the Dirac equation

$$\begin{aligned}
&\sum_{\mu=1}^n \gamma^\mu \sqrt{Q_\mu} \left(\frac{\partial}{\partial x_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{1}{2} \sum_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n \frac{x_\mu}{x_\mu^2 - x_\nu^2} \right) \Psi \\
&+ \sum_{\mu=1}^n \gamma^{n+\mu} \sqrt{Q_\mu} \left(\sum_{k=0}^{n-1} \frac{(-1)^k x_\mu^{2(n-1-k)}}{X_\mu} \frac{\partial}{\partial \psi_k} + \frac{1}{2} \sum_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n \frac{x_\nu}{x_\mu^2 - x_\nu^2} (\gamma^\nu \gamma^{n+\nu}) \right) \Psi + m\Psi = 0.
\end{aligned} \tag{13}$$

Let us use the following representation of γ -matrices: $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$,

$$\begin{aligned}
\gamma^\mu &= \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{\mu-1} \otimes \sigma_1 \otimes I \otimes \cdots \otimes I, \\
\gamma^{n+\mu} &= \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{\mu-1} \otimes \sigma_2 \otimes I \otimes \cdots \otimes I,
\end{aligned} \tag{14}$$

where I is the 2×2 identity matrix and σ_i are the Pauli matrices. In this representation, we write the 2^n components of the spinor field as $\Psi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}$ ($\epsilon_\mu = \pm 1$), and it follows that

$$\begin{aligned}
(\gamma^\mu \Psi)_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} &= \left(\prod_{\nu=1}^{\mu-1} \epsilon_\nu \right) \Psi_{\epsilon_1 \cdots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \cdots \epsilon_n}, \\
(\gamma^{n+\mu} \Psi)_{\epsilon_1 \epsilon_2 \cdots \epsilon_n} &= -i\epsilon_\mu \left(\prod_{\nu=1}^{\mu-1} \epsilon_\nu \right) \Psi_{\epsilon_1 \cdots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \cdots \epsilon_n}.
\end{aligned} \tag{15}$$

By the isometry the spinor field takes the form

$$\Psi_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x, \psi) = \hat{\Psi}_{\epsilon_1 \epsilon_2 \cdots \epsilon_n}(x) \exp \left(i \sum_{k=0}^{n-1} N_k \psi_k \right) \tag{16}$$

with arbitrary constants N_k . Substituting (15) into (13), we obtain

$$\sum_{\mu=1}^n \sqrt{Q_\mu} \left(\prod_{\rho=1}^{\mu-1} \epsilon_\rho \right) \left(\frac{\partial}{\partial x_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{1}{2} \frac{\epsilon_\mu Y_\mu}{X_\mu} + \frac{1}{2} \sum_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n \frac{1}{x_\mu - \epsilon_\mu \epsilon_\nu x_\nu} \right) \hat{\Psi}_{\epsilon_1 \dots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \dots \epsilon_n} \\ + m \hat{\Psi}_{\epsilon_1 \epsilon_2 \dots \epsilon_n} = 0, \quad (17)$$

where we have introduced the function

$$Y_\mu = \sum_{k=0}^{n-1} (-1)^k x_\mu^{2(n-1-k)} N_k, \quad (18)$$

which depends only on x_μ .

Consider now the region $x_\mu - x_\nu > 0$ for $\mu < \nu$ and $x_\mu + x_\nu > 0$. Let us define

$$\Phi_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x) = \prod_{1 \leq \mu < \nu \leq n} \frac{1}{\sqrt{x_\mu + \epsilon_\mu \epsilon_\nu x_\nu}}. \quad (19)$$

Then, one can obtain an equality

$$\frac{\Phi_{\epsilon_1 \dots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \dots \epsilon_n}(x)}{\Phi_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x)} = (-\epsilon_\mu)^{\mu-1} \left(\prod_{\rho=1}^{\mu-1} \epsilon_\rho \right) \frac{\sqrt{(-1)^{\mu-1} U_\mu}}{\prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (x_\mu - \epsilon_\mu \epsilon_\nu x_\nu)}. \quad (20)$$

Now we show that the Dirac equation allows a separation of variables by setting

$$\hat{\Psi}_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x) = \Phi_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x) \prod_{\mu=1}^n \chi_{\epsilon_\mu}^{(\mu)}(x_\mu). \quad (21)$$

It should be noticed that

$$\frac{\partial}{\partial x_\mu} \log \hat{\Psi}_{\epsilon_1 \dots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \dots \epsilon_n} = \frac{d}{dx_\mu} \log \chi_{-\epsilon_\mu}^{(\mu)} - \frac{1}{2} \sum_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n \frac{1}{x_\mu - \epsilon_\mu \epsilon_\nu x_\nu}. \quad (22)$$

By using (20) and (22), the substitution of (21) into (17) leads to

$$\sum_{\mu=1}^n \frac{P_{\epsilon_\mu}^{(\mu)}(x_\mu)}{\prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (\epsilon_\mu x_\mu - \epsilon_\nu x_\nu)} + m = 0, \quad (23)$$

where $P_{\epsilon_\mu}^{(\mu)}$ is a function of the coordinate x_μ only,

$$P_{\epsilon_\mu}^{(\mu)} = (-1)^{\mu-1} (\epsilon_\mu)^{n-\mu} \sqrt{(-1)^{\mu-1} X_\mu} \frac{1}{\chi_{\epsilon_\mu}^{(\mu)}} \left(\frac{d}{dx_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{\epsilon_\mu Y_\mu}{X_\mu} \right) \chi_{-\epsilon_\mu}^{(\mu)}. \quad (24)$$

Putting

$$Q(y) = -m y^{n-1} + \sum_{j=0}^{n-2} q_j y^j \quad (25)$$

with arbitrary constants q_j , we find

$$P_{\epsilon_\mu}^{(\mu)}(x_\mu) = Q(\epsilon_\mu x_\mu). \quad (26)$$

Thus, the functions $\chi_{\epsilon_\mu}^{(\mu)}$ satisfy the ordinary differential equations

$$\left(\frac{d}{dx_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{\epsilon_\mu Y_\mu}{X_\mu} \right) \chi_{-\epsilon_\mu}^{(\mu)} - \frac{(-1)^{\mu-1} (\epsilon_\mu)^{n-\mu} Q(\epsilon_\mu x_\mu)}{\sqrt{(-1)^{\mu-1} X_\mu}} \chi_{\epsilon_\mu}^{(\mu)} = 0. \quad (27)$$

2. D=2n+1

For the metric (2) we introduce the orthonormal basis $\{\hat{e}^a\} = \{\hat{e}^\mu, \hat{e}^{n+\mu}, \hat{e}^{2n+1}\}$ ($\mu = 1, 2, \dots, n$):

$$\hat{e}^\mu = e^\mu, \quad \hat{e}^{n+\mu} = e^{n+\mu}, \quad \hat{e}^{2n+1} = \sqrt{S} \sum_{k=0}^n A^{(k)} d\psi_k \quad (28)$$

with $S = c/A^{(n)}$. The 1-forms e^μ and $e^{n+\mu}$ are defined by (8). The dual vector fields are given by

$$\hat{e}_\mu = e_\mu, \quad \hat{e}_{n+\mu} = e_{n+\mu} + \frac{(-1)^n}{x_\mu^2 \sqrt{Q_\mu} U_\mu} \frac{\partial}{\partial \psi_n}, \quad \hat{e}_{2n+1} = \frac{1}{\sqrt{S} A^{(n)}} \frac{\partial}{\partial \psi_n} \quad (29)$$

with (9). The spin connection is calculated as [11]

$$\begin{aligned} \hat{\omega}_{\mu\nu} &= \omega_{\mu\nu}, \quad \hat{\omega}_{\mu, n+\nu} = \omega_{\mu, n+\nu} + \delta_{\mu\nu} \frac{\sqrt{S}}{x_\mu} \hat{e}^{2n+1}, \quad \hat{\omega}_{n+\mu, n+\nu} = \omega_{n+\mu, n+\nu}, \\ \hat{\omega}_{\mu, 2n+1} &= \frac{\sqrt{S}}{x_\mu} \hat{e}^{n+\mu} - \frac{\sqrt{Q_\mu}}{x_\mu} \hat{e}^{2n+1}, \quad \hat{\omega}_{n+\mu, 2n+1} = -\frac{\sqrt{S}}{x_\mu} \hat{e}^\mu. \end{aligned} \quad (30)$$

A similar calculation to the even dimensional case yields the following Dirac equation,

$$\begin{aligned} & \sum_{\mu=1}^n \gamma^\mu \sqrt{Q_\mu} \left(\frac{\partial}{\partial x_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{1}{2x_\mu} + \frac{1}{2} \sum_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n \frac{x_\mu}{x_\mu^2 - x_\nu^2} \right) \Psi \\ & + \sum_{\mu=1}^n \gamma^{n+\mu} \sqrt{Q_\mu} \left(\sum_{k=0}^{n-1} \frac{(-1)^k x_\mu^{2(n-1-k)}}{X_\mu} \frac{\partial}{\partial \psi_k} + \frac{(-1)^n}{x_\mu^2 X_\mu} \frac{\partial}{\partial \psi_n} + \frac{1}{2} \sum_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n \frac{x_\nu}{x_\mu^2 - x_\nu^2} (\gamma^\nu \gamma^{n+\nu}) \right) \Psi \\ & + \gamma^{2n+1} \sqrt{S} \left(- \sum_{\mu=1}^n \frac{1}{2x_\mu} (\gamma^\mu \gamma^{n+\mu}) + \frac{1}{c} \frac{\partial}{\partial \psi_n} \right) \Psi + m \Psi = 0. \end{aligned} \quad (31)$$

We use the representation of γ -matrices given by (14) together with

$$\gamma^{2n+1} = \sigma_3 \otimes \sigma_3 \otimes \dots \otimes \sigma_3. \quad (32)$$

Thus, the spinor field $\hat{\Psi}_{\epsilon_1 \epsilon_2 \dots \epsilon_n}$ defined by

$$\Psi_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x, \psi) = \hat{\Psi}_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x) \exp \left(i \sum_{k=0}^n N_k \psi_k \right) \quad (33)$$

satisfies the equation

$$\begin{aligned} & \sum_{\mu=1}^n \sqrt{Q_\mu} \left(\prod_{\rho=1}^{\mu-1} \epsilon_\rho \right) \left(\frac{\partial}{\partial x_\mu} + \frac{1}{2} \frac{X'_\mu}{X_\mu} + \frac{1}{2} \frac{\epsilon_\mu \hat{Y}_\mu}{X_\mu} \right. \\ & \quad \left. + \frac{1}{2x_\mu} + \frac{1}{2} \sum_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n \frac{1}{x_\mu - \epsilon_\mu \epsilon_\nu x_\nu} \right) \hat{\Psi}_{\epsilon_1 \dots \epsilon_{\mu-1} (-\epsilon_\mu) \epsilon_{\mu+1} \dots \epsilon_n} \\ & + \left(i \sqrt{S} \left(\prod_{\rho=1}^n \epsilon_\rho \right) \left(- \sum_{\mu=1}^n \frac{\epsilon_\mu}{2x_\mu} + \frac{N_n}{c} \right) + m \right) \hat{\Psi}_{\epsilon_1 \epsilon_2 \dots \epsilon_n} = 0, \end{aligned} \quad (34)$$

where

$$\hat{Y}_\mu = \sum_{k=0}^n (-1)^k x_\mu^{2(n-1-k)} N_k. \quad (35)$$

We find that the Dirac equation above allows a separation of variables

$$\hat{\Psi}_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x) = \Phi_{\epsilon_1 \epsilon_2 \dots \epsilon_n}(x) \prod_{\mu=1}^n \left(\frac{\chi_{\epsilon_\mu}^{(\mu)}(x_\mu)}{\sqrt{x_\mu}} \right) \quad (36)$$

with $\Phi_{\epsilon_1 \epsilon_2 \dots \epsilon_n}$ defined by (19). Indeed, (34) becomes

$$\sum_{\mu=1}^n \frac{P_{\epsilon_\mu}^{(\mu)}(x_\mu)}{\prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (\epsilon_\mu x_\mu - \epsilon_\nu x_\nu)} + \frac{i\sqrt{c}}{\prod_{\rho=1}^n (\epsilon_\rho x_\rho)} \left(-\sum_{\mu=1}^n \frac{\epsilon_\mu}{2x_\mu} + \frac{N_n}{c} \right) + m = 0 \quad (37)$$

with the help of (24). Let us introduce the function

$$\hat{Q}(y) = \sum_{j=-2}^{n-1} q_j y^j \quad (38)$$

where

$$q_{n-1} = -m, \quad q_{-1} = \frac{i}{2}(-1)^{n-1}\sqrt{c}, \quad q_{-2} = \frac{i}{\sqrt{c}}(-1)^n N_n. \quad (39)$$

Using the identities

$$\sum_{\mu=1}^n \frac{1}{y_\mu^2 \prod_{\substack{\nu \\ (\nu \neq \mu)}} (y_\mu - y_\nu)} = \frac{(-1)^{n-1}}{\prod_{\mu=1}^n y_\mu} \sum_{\nu=1}^n \frac{1}{y_\nu} \quad (40)$$

$$\sum_{\mu=1}^n \frac{1}{y_\mu \prod_{\substack{\nu \\ (\nu \neq \mu)}} (y_\mu - y_\nu)} = \frac{(-1)^{n-1}}{\prod_{\mu=1}^n y_\mu} \quad (41)$$

we can confirm that the functions $\chi_{\epsilon_\mu}^{(\mu)}$ satisfy the ordinary differential equations (27) by the replacements $Y_\mu \rightarrow \hat{Y}_\mu$ and $Q(\epsilon_\mu x_\mu) \rightarrow \hat{Q}(\epsilon_\mu x_\mu)$.

We have shown the separation of variables of Dirac equations in general Kerr-NUT-de Sitter spacetimes. An interesting problem is to investigate the origin of separability. In the case of geodesic Hamilton-Jacobi equations and Klein-Gordon equations we know that the existence of separable coordinates comes from that of a rank-2 closed conformal Killing-Yano tensor. We can also construct the first order differential operators from the closed conformal Killing-Yano tensor which commute with Dirac operators [12, 13, 14]. However, we have no clear answer of the separability of Dirac equations. As another problem we can study eigenvalues of Dirac operators on Sasaki-Einstein manifolds. Indeed, as shown in [1, 15, 16, 17], the BPS limit of odd-dimensional Kerr-NUT-de Sitter metrics leads to Sasaki-Einstein metrics. Especially, the five-dimensional metrics are important from the point of view of AdS/CFT correspondence.

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